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FUNDAMENTAL SOLUTIONS FOR LINEAR VISCOELASTIC CONTINUA

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Abstract—The full set of fundamental solutions for the linear viscoelastic problem is derived from the relevant elastic fundamental solutions using the well-known so-called "correspondence principle". The fundamental solutions are given for the 3D-continuum, the 2D-plane strain and the 2D-plane stress problem using a wide range of linear viscoelastic constitutive law models.

Finally, the possible fields of application of the determined fundamental solutions are given using symmetric BIE formulations in space and time.

1. INTRODUCTION

The problem of determining the response of a viscoelastic body under given time-dependent loads has been tackled with a number of methods depending on the numerical technique and the representation of the constitutive model adopted.

After the first formulations of the viscoelastic problem by Boltzmann (1874) (for a 3D isotropic continuum) and by Volterra (1909, 1913) (for anisotropic solids), the correspondence principle of Alfrey (1944, 1945) gave, through the Laplace transform technique (Doetsch, 1956; Ghizzetti and Ossicini, 1971), a formal method widely and successfully adopted by Lee (1955), Mandel (1955) and others (Tsien, 1950; Radok, 1957; Lee, 1958; Naghdi and Orthwein, 1960; Tao, 1963; Rogers and Lee, 1964; Herrera and Gurtin, 1964; Graham, 1968; Rizzo and Shippy, 1971; Carpenter, 1972; Graham and Sabin, 1973) to derive the viscoelastic solution of a problem when the corresponding elastic solution is available in closed form.

Then, in connection with the evolution of the numerical solution techniques for elastic problems, the step by step time integration together with the finite element space-discretization methods were adopted by a number of authors (White, 1958; Zienkiewicz *et al.*, 1968; Webber, 1969; Taylor *et al.*, 1970; Cameron and McKee, 1981).

In the same period another trend of studies [mainly developed by Biot (1955), Gurtin (1963), Schapery (1964), Leitman (1966), Christensen (1968), Tonti (1973), Reddy (1976) and Reddy and Rasmussen (1982)] was devoted to the formulation of the same problem in variational terms in space and time.

More recently, in connection with the development of the theory of the boundary integral equations (BIE) for elastic continua (Brebbia *et al.*, 1984), another approach was adopted for viscoelastic problems using the correspondence principle (Banerjee and Butterfield, 1981; Kusama and Mitsui, 1982; Shinokawa *et al.*, 1985; Wolf and Darbre, 1985; Sun and Hsiao, 1985; Tanaka, 1985; Carini and Gioda, 1986).

In the context of the BIE method, recent contributions for elastic (Sirtori, 1979; Polizzotto, 1988a,b) and elastoplastic (Maier and Polizzotto, 1987; Polizzotto, 1988a,b) continua and for the steady-state heat conduction problem (Costabel, 1987) allowed, by the variational methods adopted for the viscoelastic problems, to attain even under a BIE approach a variational formulation in space and time of linear time dependent problems like transient heat conduction (Carini *et al.*, 1991a,b), elastodynamics (Maier *et al.*, 1991), viscoelasticity (Carini *et al.*, 1991a,b), etc.

In the above contributions, Gebbia's (1891, 1902, 1904) fundamental solution for a unit concentrated displacement discontinuity (besides the well-known solution for a unit concentrated force due to Kelvin) was required to find the BIE elastic symmetric formulation; analogously, the fundamental solutions for a unit concentrated displacement discontinuity and for a unit concentrated deformation were required for the corresponding elastoplastic symmetric formulation.

In this context the aim of the present paper is to find, in the viscoelastic range, the fundamental solutions corresponding to the above elastic solutions as a contribution to the development of variational formulations of the viscoelasticity in space and time by BIE approach. Moreover, these solutions are also going to prove their usefulness for the variational, formulations of visco-elasto–plastic continua.

In Section 2, after a brief summary of the classical representations of the hereditary linear viscoelastic constitutive laws in integral and differential form, the recursive formula for the coefficients of the differential form for the generalized Kelvin and Maxwell models are derived (Appendix A).

Section 3 is devoted to the description of the correlations between the elastic fundamental solutions, which are collected, from a scattered literature, in a compact form in Appendix B.

Finally, in Section 5, using the correspondence principle (described in Section 4), the viscoelastic fundamental solutions are derived using the differential form of representation of the constitutive laws.

2. HEREDITARY LINEAR VISCOELASTIC CONSTITUTIVE LAW

The two classical representations of the hereditary linear viscoelastic constitutive law (Gurtin and Sternberg, 1962; Mandel, 1966; Christensen, 1971) used in the following, are summarized here:

(a) *Integral form*

A classical representation at time t of the above law in the integral form is the following:

$$\sigma_{ij}(\mathbf{x},t) = H_{ijhk}(\mathbf{x},t)\cdot\varepsilon_{hk}(\mathbf{x},0)+\int_0^t H_{ijhk}(\mathbf{x},t-\tau)\,\mathrm{d}\varepsilon_{hk}(\mathbf{x},\tau), \tag{1}$$

where σ_{ij}, ε_{hk}, H_{ijhk} are the stress tensor, strain tensor and relaxation viscous kernel tensor, respectively, τ being the integration variable. An equivalent form, derived from (1) by integration by parts is:

$$\sigma_{ij}(\mathbf{x},t) = H_{ijhk}(\mathbf{x},0)\varepsilon_{hk}(\mathbf{x},t)+\int_0^t \frac{\partial H_{ijhk}(\mathbf{x},t-\tau)}{\partial(t-\tau)}\varepsilon_{hk}(\mathbf{x},\tau)\,\mathrm{d}\tau. \tag{1a}$$

The inverse constitutive law has the following corresponding forms:

$$\varepsilon_{ij}(\mathbf{x},t) = K_{ijhk}(\mathbf{x},t)\sigma_{hk}(\mathbf{x},0)+\int_0^t K_{ijhk}(\mathbf{x},t-\tau)\,\mathrm{d}\sigma_{hk}(\mathbf{x},\tau), \tag{2}$$

$$\varepsilon_{ij}(\mathbf{x},t) = K_{ijhk}(\mathbf{x},0)\sigma_{hk}(\mathbf{x},t)+\int_0^t \frac{\partial K_{ijhk}(\mathbf{x},t-\tau)}{\partial(t-\tau)}\sigma_{hk}(\mathbf{x},\tau)\,\mathrm{d}\tau, \tag{2a}$$

where K_{ijhk} is the creep viscous kernel tensor.

The symmetry of the stress and strain tensors implies:

$$\begin{cases} H_{ijhk}(\mathbf{x},t) = H_{jihk}(\mathbf{x},t) = H_{ijkh}(\mathbf{x},t), \\ K_{ijhk}(\mathbf{x},t) = K_{jihk}(\mathbf{x},t) = K_{ijkh}(\mathbf{x},t), \end{cases} \tag{3}$$

while for isotropic materials the following expression of the relaxation kernel $H_{ijhk}(\mathbf{x},t)$ holds:

$$H_{ijhk}(\mathbf{x},t) = \tfrac{1}{3}[H_2(\mathbf{x},t) - H_1(\mathbf{x},t)]\delta_{ij}\delta_{hk} + \tfrac{1}{2}H_1(\mathbf{x},t)(\delta_{ih}\delta_{jk} + \delta_{ik}\delta_{jh}), \tag{4}$$

$H_1(\mathbf{x},t)$ and $H_2(\mathbf{x},t)$ being the so-called tangential and volumetric relaxation functions, respectively.

Using the stress and strain deviators s_{ij}, e_{ij},

$$\begin{cases} s_{ij} = \sigma_{ij} - \tfrac{1}{3}\delta_{ij}\sigma_{kk}, \\ e_{ij} = \varepsilon_{ij} - \tfrac{1}{3}\delta_{ij}\varepsilon_{kk}, \end{cases} \tag{5}$$

the above constitutive law for isotropic materials takes the direct form

$$\begin{cases} s_{ij}(\mathbf{x},t) = H_1(\mathbf{x},t)e_{ij}(\mathbf{x},0) + \displaystyle\int_0^t H_1(\mathbf{x},t-\tau)\,\mathrm{d}e_{ij}(\mathbf{x},\tau), \\ \sigma_{kk}(\mathbf{x},t) = H_2(\mathbf{x},t)\varepsilon_{kk}(\mathbf{x},0) + \displaystyle\int_0^t H_2(\mathbf{x},t-\tau)\,\mathrm{d}\varepsilon_{kk}(\mathbf{x},\tau), \end{cases} \tag{6a,b}$$

or the inverse form

$$\begin{cases} e_{ij}(\mathbf{x},t) = K_1(\mathbf{x},t)s_{ij}(\mathbf{x},0) + \displaystyle\int_0^t K_1(\mathbf{x},t-\tau)\,\mathrm{d}s_{ij}(\mathbf{x},\tau), \\ \varepsilon_{kk}(\mathbf{x},t) = K_2(\mathbf{x},t)\sigma_{kk}(\mathbf{x},0) + \displaystyle\int_0^t K_2(\mathbf{x},t-\tau)\,\mathrm{d}\sigma_{kk}(\mathbf{x},\tau), \end{cases} \tag{7a,b}$$

where $K_1(\mathbf{x},t)$ and $K_2(\mathbf{x},t)$ are the so-called tangential and volumetric creep functions corresponding to the analogous H_1 and H_2 functions given above.

For homogeneous materials the relaxation (H_1, H_2) and creep (K_1, K_2) functions are space independent.

For isotropic materials the following have to be added to the above symmetry relations [eqn (3)]:

$$H_{ijhk} = H_{hkij}, \quad K_{ijhk} = K_{hkij}. \tag{3a}$$

(b) *Differential form*

A classical representation of the hereditary linear viscoelastic constitutive law in differential form is the following:

$$P_{ijhk}(D)\sigma_{hk}(\mathbf{x},t) = Q_{ijhk}(D)\varepsilon_{hk}(\mathbf{x},t), \tag{8}$$

where, using the notation adopted by Alfrey (1945), $P_{ijhk}(D)$ and $Q_{ijhk}(D)$ are tensors of linear differential operators, i.e.:

$$P_{ijhk}(D) = \sum_{r=0}^{n}(a_r)_{ijhk}D^r; \; Q_{ijhk}(D) = \sum_{r=0}^{n}(b_r)_{ijhk}D^r, \tag{9}$$

where the components of the tensors $(a_r)_{ijhk}$ and $(b_r)_{ijhk}$ are real constants and D^k is the operator "kth derivative", i.e.:

$$D^k f = \frac{d^k f}{dt^k};\ D^0 f = f. \tag{10}$$

For isotropic materials, eqn (8) specializes in the following two relations to:

$$\begin{cases} P_1(D)s_{ij}(\mathbf{x},t) = Q_1(D)e_{ij}(\mathbf{x},t), \\ P_2(D)\sigma_{kk}(\mathbf{x},t) = Q_2(D)\varepsilon_{kk}(\mathbf{x},t), \end{cases} \tag{11a,b}$$

where P_1, P_2 and Q_1, Q_2 are linear differential operators analogous to those defined in (9).

(c) *Connections between the integral and differential forms*

Using the Laplace transform, eqn (8) gives:

$$P_{ijhk}(q)\bar{\sigma}_{hk}(\mathbf{x},q) = Q_{ijhk}(q)\bar{\varepsilon}_{hk}(\mathbf{x},q), \tag{12}$$

where $\bar{\sigma}_{ij}(\mathbf{x},q)$ and $\bar{\varepsilon}_{ij}(\mathbf{x},q)$ are the stress and strain tensor t transforms, q being the transformation parameter, while:

$$P_{ijhk}(q) = \sum_{r=0}^{n} (a_r)_{ijhk} q^r; \qquad Q_{ijhk}(q) = \sum_{r=0}^{n} (b_r)_{ijhk} q^r. \tag{13}$$

Using the Laplace transform, eqns (1) and (2) give:

$$\begin{cases} \bar{\sigma}_{ij}(\mathbf{x},q) = q\bar{\varepsilon}_{hk}(\mathbf{x},q)\bar{H}_{ijhk}(\mathbf{x},q), \\ \bar{\varepsilon}_{ij}(\mathbf{x},q) = q\bar{\sigma}_{hk}(\mathbf{x},q)\bar{K}_{ijhk}(\mathbf{x},q), \end{cases} \tag{14}$$

where $\bar{H}_{ijhk}(\mathbf{x},q)$, $\bar{K}_{ijhk}(\mathbf{x},q)$ are the Laplace t transforms of H_{ijhk}, K_{ijhk}. By comparing eqn (12) with eqns (14), the following relations are derived:

$$\bar{H}_{ijhk}(\mathbf{x},q) = \frac{1}{q} Q_{ijhk}(\mathbf{x},q) P^{-1}_{ijhk}(\mathbf{x},q),$$

$$\bar{K}_{ijhk}(\mathbf{x},q) = \frac{1}{q} P_{ijhk}(\mathbf{x},q) Q^{-1}_{ijhk}(\mathbf{x},q). \tag{15}$$

These well-known relations allow the integral form of the viscoelastic constitutive law to be derived from the differential form.

(d) *Synthesis of various viscoelastic constitutive models*

Table 1 gives a synthetic picture of all the coefficients of the constitutive differential form written, for different uniaxial viscoelastic models, in the form:

$$\sum_{i=0}^{n} a_i \sigma^{(i)} = \sum_{i=0}^{n} b_i \varepsilon^{(i)}, \tag{16}$$

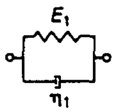
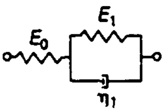
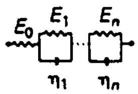
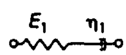
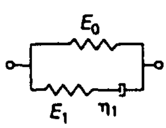
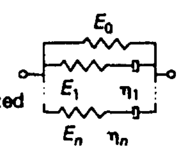
where $2n+1$ is the number of parameters defining the model and (i) means ith derivative.

The recursive formula given for the generalized Kelvin and Maxwell models are derived in Appendix A.

For a multiaxial isotropic viscoelastic behaviour, two independent rheological models governing the development of creep strains can be introduced involving the hydrostatic part of the stress σ_{kk} and of the strain ε_{kk} tensor and the deviatoric part of the stress s_{ij} and of the strain e_{ij} tensor, respectively.

The equations governing the behaviour of the two models can be written in the following form analogous to eqn (16) where m is the number of parameters defining the

Table 1. Coefficients a_i, b_i of the hereditary linear viscoelastic constitutive law written in the differential form eqn (16) and relevant relaxation and creep kernels $H(t), K(t)$ of the integral form eqns (1), (2) for different uniaxial viscoelastic models ($2n+1$ = number of parameters of the model)

Model type		Constitutive law: Differential form a_i , b_i	Constitutive law: Integral form $H(t)$, $K(t)$
Kelvin		$a_0 = 1;\ a_1 = 0$ $b_0 = E_1;\ b_1 = \eta_1$	$H(t) = E_1 + \eta_1\,\delta_t$ $K(t) = \frac{1}{E_1}\left[1 - e^{-\frac{E_1}{\eta_1}t}\right]$
Kelvin–Voigt		$a_0 = \frac{E_0+E_1}{E_1};\ a_1 = \frac{\eta_1}{E_0}$ $b_0 = E_1\ ;\ b_1 = \eta_1$	$H(t) = E_0 - \frac{E_0^2}{E_0+E_1}\left[1 - e^{-(E_0+E_1)t/\eta_1}\right]$ $K(t) = \frac{1}{E_0} + \frac{1}{E_1}\left[1 - e^{-E_1 t/\eta_1}\right]$
Kelvin–generalized		See the recursive formulae: (A12), (A13), (A14)	$K(t) = \frac{1}{E_0} + \sum_i^n{}_1 \frac{1}{E_1}\left[1 - e^{-\frac{E_i}{\eta_i}t}\right]$
Maxwell		$a_0 = \frac{1}{\eta_1}\ ;\ a_1 = \frac{1}{E_1}$ $b_0 = 0\ \ ;\ b_1 = 1$	$H(t) = E_1 e^{-\frac{E_1}{\eta_1}t}$ $K(t) = \frac{1}{E_1} + \frac{t}{\eta_1}$
Zener		$a_0 = \frac{1}{\eta_1}\ ;\ a_1 = \frac{1}{E_1}$ $b_0 = E_0/\eta_1\ ;\ b_1 = \frac{E_0+E_1}{E_1}$	$H(t) = (E_0+E_1) - E_1\left[1 - e^{-E_1 t/\eta_1}\right]$ $K(t) = \frac{1}{E_0+E_1} + \frac{E_1}{E_0\ (E_0+E_1)}\left[1 - e^{-\frac{E_0 E_1}{(E_0+E_1)\eta_1}t}\right]$
Maxwell generalized		See the recursive formulae: (A16), (A17), (A18)	$H(t) = E_0 - \sum_i^n{}_1 E_i\left[1 - e^{-\frac{E_i}{\eta_i}t}\right]$

idrostatic part of the stress and strain, while n is the number of parameters defining the corresponding deviatoric part,

$$\sum_{i=0}^{m} a_i^h \sigma_{kk}^{(i)} = \sum_{i=0}^{m} b_i^h \varepsilon_{kk}^{(i)}\ ;\ \sum_{i=0}^{n} a_i^d s_{hk}^{(i)} = \sum_{i=0}^{n} b_i^d e_{hk}^{(i)}. \tag{17}$$

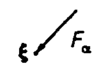
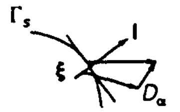
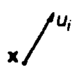
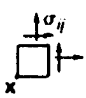
where a_i^h, a_i^d and b_i^h, b_i^d are given by the coefficients a_i, b_i in Table 1, respectively.

3. FUNDAMENTAL SOLUTIONS IN LINEAR ELASTICITY

A synthetic picture of the fundamental solutions of elastic homogeneous isotropic and unbounded continua for displacements, tractions and stresses given by a unit concentrated force F_α, a unit concentrated discontinuity of displacement D_α, and a unit concentrated deformation $\Theta_{\alpha\beta}$, may be organized in Table 2 (Maier *et al.*, 1987).

In Table 2 **l** is the unit vector normal to the surface Γ_ν where the concentrated displacement discontinuity D_α is imposed in the "load point" ξ and **n** is the unit vector normal to the surface Γ_e where the effect (traction vector **p**) of a source action is evaluated in the "field point" **x**; indices i, j are representative of components of effects in **x**, while α, β are indices representative of components of causes in ξ. Analogously the upper indices represent the effect and the cause, respectively, taking into account that the dual variables of the cause instead of the cause variables are indicated (i.e. instead of the cause variables $F_\alpha, D_\alpha, \Theta_{\alpha\beta}$, the dual variables in the virtual work sense u, p, σ are used).

Table 2. *Elastic* fundamental solutions for displacements tractions and stresses in x, given by a unit concentrated force F_α, a unit concentrated discontinuity of displacement D_α, and a unit concentrated deformation $\Theta_{\alpha\beta}$, all applied in ξ

		Source in ξ		
		Unit concentrated force F_α 	Unit conc. displ. discont. D_α 	Unit conc. strain discont. $\Theta_{\alpha\beta}$ 
Effect in x	Displacement $u_i(x)$ 	$G^{uu}_{i\alpha}(x,\xi)$	$G^{up}_{i\alpha}(x,\xi,l)$	$G^{u\sigma}_{i\alpha\beta}(x,\xi)$
	Traction $P_i(x)$ 	$G^{pu}_{i\alpha}(x,\xi,n)$	$G^{pp}_{i\alpha}(x,\xi,n,l)$	$G^{p\sigma}_{i\alpha\beta}(x,\xi,n)$
	Stress $\sigma_{ij}(x)$ 	$G^{\sigma u}_{ij\alpha}(x,\xi)$	$G^{\sigma p}_{ij\alpha}(x,\xi,l)$	$G^{\sigma\sigma}_{ij\alpha\beta}(x,\xi)$

This notation allows us to have in Table 2 a symmetric index formal representation of symmetric fundamental solutions. In particular, due to Betti's theorem (for $x \neq \xi$):

$$G^{uu}_{i\alpha}(x,\xi) = G^{uu}_{\alpha i}(\xi,x), \tag{18a}$$

$$G^{pp}_{i\alpha}(x,\xi,n,l) = G^{pp}_{\alpha i}(\xi,x,l,n), \tag{18b}$$

$$G^{\sigma\sigma}_{ij\alpha\beta}(x,\xi) = G^{\sigma\sigma}_{\alpha\beta ij}(\xi,x), \tag{18c}$$

$$G^{pu}_{i\alpha}(x,\xi,n) = G^{up}_{\alpha i}(\xi,x,l), \tag{18d}$$

$$G^{\sigma u}_{ij\alpha}(x,\xi) = G^{u\sigma}_{\alpha ij}(\xi,x), \tag{18e}$$

$$G^{\sigma p}_{ij\alpha}(x,\xi,l) = G^{p\sigma}_{\alpha ij}(\xi,x,n), \tag{18f}$$

where in the second members the rôles of the load and field points ξ, x together with the unit normal vectors l, u are mutually exchanged (in particular in eqns (18d), (18f) the components of the unit vectors l, u coincide).

In Appendix B the full expressions of all the above fundamental solutions in linear elasticity together with their correlations are briefly collected.

4. CORRESPONDENCE PRINCIPLE

The so-called "correspondence principle" (Alfrey, 1944) based on the use of the Laplace transform (Doetsch, 1956; Ghizzetti and Ossicini, 1971) permits us to write the above constitutive equations (6) or (11) for the homogeneous material case in a form analogous to that usually adopted for the linear elastic law:

Table 3. Values of r and s of eqn (22a,b) depending on the number m and n of parameters defining the idrostatic and the deviatoric part of the constitutive law respectively

	i	r	s
$m<n$	$0 \leqslant i \leqslant m$	0	i
	$m+1 \leqslant i \leqslant n$	0	m
	$n+1 \leqslant i \leqslant n+m$	$i-n$	m
$m=n$	$0 \leqslant i \leqslant n$	0	i
	$n+1 \leqslant i \leqslant 2n$	$i-n$	n
$m>n$	$0 \leqslant i \leqslant n$	0	i
	$n+1 \leqslant i \leqslant m$	$i-n$	i
	$m+1 \leqslant i \leqslant n+m$	$i-n$	m

$$\bar{\sigma}_{kk}(\mathbf{x},q) = \frac{\bar{E}_v(q)}{1-2\bar{v}_v(q)}\bar{\varepsilon}_{kk}(\mathbf{x},q) = q\bar{H}_2(q)\bar{\varepsilon}_{kk}(\mathbf{x},q) = 3\bar{K}_v(q)\bar{\varepsilon}_{kk}(\mathbf{x},q), \tag{19a}$$

$$\bar{s}_{ij}(\mathbf{x},q) = \frac{\bar{E}_v(q)}{1+\bar{v}_v(q)}\bar{e}_{ij}(\mathbf{x},q) = q\bar{H}_1(q)\bar{e}_{ij}(\mathbf{x},q) = 2\bar{G}_v(q)\bar{e}_{ij}(\mathbf{x},q). \tag{19b}$$

In the above equations, q is the complex variable, a superposed bar denotes the transformed variables and the following expressions hold for the equivalent material parameters (Carini and Gioda, 1986):

$$\bar{G}_v(q) = \frac{q}{2}\bar{H}_1(q) = \frac{1}{2}\frac{\sum_{i=0}^{n} b_i^d q^i}{\sum_{i=0}^{n} a_i^d q^i}; \quad \bar{K}_v(q) = \frac{p}{3}\bar{H}_2(q) = \frac{1}{3}\frac{\sum_{i=0}^{m} b_i^h q^i}{\sum_{i=0}^{m} a_i^h q^i}, \tag{20a,b}$$

$$\bar{v}_v(q) = \frac{\bar{H}_2(q)-\bar{H}_1(q)}{\bar{H}_1(q)+2\bar{H}_2(q)} = \frac{\sum_{i=0}^{m+n} A_i q^i}{\sum_{i=0}^{m+n} B_i q^i}, \tag{21}$$

where q^i is the ith power of q.

The coefficients A_i and B_i in eqns (21) are defined as follows:

$$A_i = \sum_{j=r}^{s} (b_j^h a_{i-j}^d - a_j^h b_{i-j}^d); \quad B_i = \sum_{j=r}^{s} (2b_j^h a_{i-j}^d + a_j^h b_{i-j}^d), \tag{22a,b}$$

where r and s are given in Table 3.

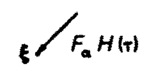
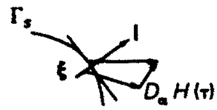
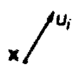
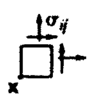
The correspondence principle can be applied to the solution of visco-elastic stress analysis problems, if in addition to the constitutive relations the equilibrium and compatibility equations, and their boundary conditions, can be transformed according to Laplace.

In the following section, the visco-elastic fundamental solutions are derived using the above correspondence principle and the differential form of the constitutive laws; the adoption of the integral form, in fact, does not lead to an explicit form of the visco-elastic solution in the absence of the explicit knowledge of Kernels $H_1(t)$, $H_2(t)$.

5. FUNDAMENTAL VISCO-ELASTIC SOLUTIONS

A synthetic picture of the visco-elastic fundamental solutions for displacements, tractions and stresses given by a unit concentrated force $F_z H(\tau)$ (where $H(\tau)$ is the Heaviside unit step function), a unit concentrated discontinuity of displacement $D_z^i H(\tau)$, and a unit

Table 4. *Viscoelastic* fundamental solutions for displacements, tractions and stresses in $\mathbf{x}$ at time t given by a unit concentrated force $F_\alpha H(t)$, a unit concentrated discontinuity $D_\alpha H(t)$, and a unit concentrated deformation $\Theta_{\alpha\beta} H(t)$, all applied in ξ at time τ

		Source in ξ, τ		
		Unit concentrated force $F_\alpha H(\tau)$	Unit conc. displ. discont. $D_\alpha H(\tau)$	Unit conc. strain discont. $\Theta_{\alpha\beta} H(\tau)$
				
Effect in $\mathbf{x}, t$: Displacement $u_i(\mathbf{x},t)$		$V^{uu}_{i\alpha}(\mathbf{x},\xi,t-\tau)$	$V^{up}_{i\alpha}(\mathbf{x},\xi,\mathbf{l},t-\tau)$	$V^{u\sigma}_{i\alpha\beta}(\mathbf{x},\xi,t-\tau)$
Effect in $\mathbf{x}, t$: Traction $P_i(\mathbf{x},t)$		$V^{pu}_{i\alpha}(\mathbf{x},\xi,\mathbf{n},t-\tau)$	$V^{pp}_{i\alpha}(\mathbf{x},\xi,\mathbf{n},\mathbf{l},t-\tau)$	$V^{p\sigma}_{i\alpha\beta}(\mathbf{x},\xi,\mathbf{n},t-\tau)$
Effect in $\mathbf{x}, t$: Stress $\sigma_{ij}(\mathbf{x},t)$		$V^{\sigma u}_{ij\alpha}(\mathbf{x},\xi,t-\tau)$	$V^{\sigma p}_{ij\alpha}(\mathbf{x},\xi,\mathbf{l},t-\tau)$	$V^{\sigma\sigma}_{ij\alpha\beta}(\mathbf{x},\xi,t-\tau)$

concentrated deformation $\Theta_{\alpha\beta} H(\tau)$, may be organized, as for the elastic case of Section 3 in the following Table 4 form.

In Table 4 the symbols have the same meaning as in Table 2 for the elastic case; in particular $V^{uu}_{i\alpha}(\mathbf{x},\xi,t-\tau)$, $V^{pu}_{i\alpha}(\mathbf{x},\xi,\mathbf{n},t-\tau)$ and $V^{\sigma u}_{ij\alpha}(\mathbf{x},\xi,t-\tau)$ are the effects (in a given material point $\mathbf{x}$ at time t) of the application of load $F_\alpha H(\tau)$ (with $\tau \leqslant t$) in terms of the displacement component $u_i(\mathbf{x},t)$, of tractions $p_i(\mathbf{x},t)$ on the surface of normal $\mathbf{n}$ and of stress components $\sigma_{ij}(\mathbf{x},t)$ respectively, i.e. (using Greek and Latin symbols for "cause" and "effect" variables respectively):

$$\begin{cases} V^{uu}_{i\alpha}(\mathbf{x},\xi,t-\tau)F_\alpha H(\tau) &= u_i(\mathbf{x},t), \\ V^{pu}_{i\alpha}(\mathbf{x},\xi,n_i,t-\tau)F_\alpha H(\tau) &= \sigma_{ij}(\mathbf{x},t)n_j = p_i(\mathbf{x},t), \\ V^{\sigma u}_{ij\alpha}(\mathbf{x},\xi,t-\tau)F_\alpha H(\tau) &= \sigma_{ij}(\mathbf{x},t). \end{cases} \tag{23}$$

$V^{up}_{i\alpha}(\mathbf{x},\xi,\mathbf{l},t-\tau)$, $V^{pp}_{i\alpha}(\mathbf{x},\xi,\mathbf{n},\mathbf{l},t-\tau)$ and $V^{\sigma p}_{ij\alpha}(\mathbf{x},\xi,\mathbf{l},t-\tau)$ are the effects (in a given material point $\mathbf{x}$ at time t) in terms of the displacement component $u_i(\mathbf{x},t)$, of tractions $p_i(\mathbf{x},t)$ on the surface of normal $\mathbf{n}$ and of stress component $\sigma_{ij}(\mathbf{x},t)$ respectively, of the application of a unit concentrated discontinuity of displacement $D^l_\alpha H(\tau)$, i.e.

$$\begin{cases} V^{up}_{i\alpha}(\mathbf{x},\xi,\mathbf{l},t-\tau)D^l_\alpha H(\tau) &= u_i(\mathbf{x},t), \\ V^{pp}_{i\alpha}(\mathbf{x},\xi,\mathbf{n},\mathbf{l},t-\tau)D^l_\alpha H(\tau) &= \sigma_{ij}(\mathbf{x},t)n_j = p_i(\mathbf{x},t), \\ V^{\sigma p}_{ij\alpha}(\mathbf{x},\xi,\mathbf{l},t-\tau)D^l_\alpha H(\tau) &= \sigma_{ij}(\mathbf{x},t). \end{cases} \tag{24}$$

Finally $V^{u\sigma}_{i\alpha\beta}(\mathbf{x},\xi,t-\tau)$, $V^{p\sigma}_{i\alpha\beta}(\mathbf{x},\xi,\mathbf{n},t-\tau)$ and $V^{\sigma\sigma}_{ij\alpha\beta}(\mathbf{x},\xi,t-\tau)$ are the effects (in a given material point $\mathbf{x}$ at time t) in terms of displacement component $u_i(\mathbf{x},t)$, of tractions $p_i(\mathbf{x},t)$ on the surface of normal $\mathbf{n}$ and of stress component $\sigma_{ij}(\mathbf{x},t)$ respectively of the application of a unit concentrated deformation $\Theta_{\alpha\beta} H(\tau)$, i.e.

$$\left\{\begin{array}{ll} V^{u\sigma}_{i\alpha\beta}(\mathbf{x},\boldsymbol{\xi},t-\tau)\Theta_{\alpha\beta}H(\tau) & = u_i(\mathbf{x},t), \\ V^{p\sigma}_{i\alpha\beta}(\mathbf{x},\boldsymbol{\xi},\mathbf{n},t-\tau)\Theta_{\alpha\beta}H(\tau) & = \sigma_{ij}(\mathbf{x},t)n_j = p_i(\mathbf{x},t), \\ V^{\sigma\sigma}_{ij\alpha\beta}(\mathbf{x},\boldsymbol{\xi},t-\tau)\Theta_{\alpha\beta}H(\tau) & = \sigma_{ij}(\mathbf{x},t). \end{array}\right. \tag{25}$$

Using the reciprocity theorem of viscoelasticity (Gurtin and Sternberg, 1962; Christensen, 1971) the following relations can be easily stated (for $\mathbf{x} \neq \boldsymbol{\xi}$):

$$V^{uu}_{i\alpha}(\mathbf{x},\boldsymbol{\xi};\,t-\tau) = V^{uu}_{\alpha i}(\boldsymbol{\xi},\mathbf{x};\,t-\tau), \tag{26a}$$

$$V^{pp}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{n},\mathbf{l};\,t-\tau) = V^{pp}_{\alpha i}(\boldsymbol{\xi},\mathbf{x},\mathbf{l},\mathbf{n};\,t-\tau), \tag{26b}$$

$$V^{\sigma\sigma}_{ij\alpha\beta}(\mathbf{x},\boldsymbol{\xi};\,t-\tau) = V^{\sigma\sigma}_{\alpha\beta ij}(\boldsymbol{\xi},\mathbf{x};\,t-\tau), \tag{26c}$$

$$V^{pu}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{n};\,t-\tau) = V^{up}_{\alpha i}(\boldsymbol{\xi},\mathbf{x},\mathbf{l};\,t-\tau), \tag{26d}$$

$$V^{\sigma u}_{ij\alpha}(\mathbf{x},\boldsymbol{\xi};\,t-\tau) = V^{u\sigma}_{\alpha ij}(\boldsymbol{\xi},\mathbf{x};\,t-\tau), \tag{26e}$$

$$V^{\sigma p}_{ij\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{l};\,t-\tau) = V^{p\sigma}_{\alpha ij}(\boldsymbol{\xi},\mathbf{x},\mathbf{n};\,t-\tau), \tag{26f}$$

where the same remarks made for the corresponding eqns (18a)–(18f) on the exchanged rôles of the load and field points and normal unit vectors, still hold.

5.1. *Viscoelastic solution* $V^{uu}_{i\alpha}(\mathbf{x},\boldsymbol{\xi};\,t-\tau)$

(a) *Elastic solution dependent form.* The Laplace transform, with respect to time t, $\bar{G}^{uu}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},q)$ of $G^{uu}_{i\alpha}(\mathbf{x},\boldsymbol{\xi})$ is [using eqn (B6)]:

$$\bar{G}^{uu}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},q) = \frac{1}{q}G^{uu}_{i\alpha}(\mathbf{x},\boldsymbol{\xi}) = \frac{1}{q}\left\{\frac{1}{16\pi(1-\nu)Gr}[(3-4\nu)\delta_{i\alpha}+r_{/i}r_{/\alpha}]\right\}. \tag{27}$$

Considering the elastic parameters of $G^{uu}_{i\alpha}(\mathbf{x},\boldsymbol{\xi})$ as functions of time and taking into account the correspondence principle, the Laplace transform $\bar{V}^{uu}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},q)$ of $V^{uu}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},t)$ is given by:

$$\bar{V}^{uu}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},q) = \frac{1}{q}G^{uu}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\bar{v}_v,\bar{G}_v). \tag{28}$$

Combining eqns (28) and (27), the following relationship between $\bar{V}^{uu}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},q)$ and $\bar{G}^{uu}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},q)$ is derived:

$$\bar{V}^{uu}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},q) = \bar{G}^{uu}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},q)\frac{(1-\nu)G[(3-4\bar{v}_v)\delta_{i\alpha}+r_{/i}r_{/\alpha}]}{(1-\bar{v}_v)\bar{G}_v[(3-4\nu)\delta_{i\alpha}+r_{/i}r_{/\alpha}]}. \tag{29}$$

Now, substituting $\bar{G}_v e \bar{v}_v$ given by eqns (20a) and (21) in eqn (29) and introducing the coefficients:

$$h_1 = \frac{(1-\nu)G[3\delta_{i\alpha}+r_{/i}r_{/\alpha}]}{(3-4\nu)\delta_{i\alpha}+r_{/i}r_{/\alpha}};\; h_2 = \frac{-4(1-\nu)G\delta_{i\alpha}}{(3-4\nu)\delta_{i\alpha}+r_{/i}r_{/\alpha}}, \tag{30a,b}$$

eqn (29) can be written as:

$$\bar{V}^{uu}_{i\alpha}(\mathbf{x},\xi,q) = \bar{G}^{uu}_{i\alpha}(\mathbf{x},\xi,q)\frac{h_1+h_2\bar{v}_v}{(1-\bar{v}_v)\bar{G}_v} = \bar{G}^{uu}_{i\alpha}(\mathbf{x},\xi,q)\frac{h_1+h_2\left(\sum_{i=0}^{m+n}A_iq^i\Big/\sum_{i=0}^{m+n}B_iq^i\right)}{\left(1-\dfrac{\sum_{i=0}^{m+n}A_iq^i}{\sum_{i=0}^{m+n}B_iq^i}\right)\dfrac{\sum_{i=0}^{n}b_i^dq^i}{\sum_{i=0}^{n}a_i^dq^i}}. \tag{31}$$

Let us now rewrite eqn (31) in the following more compact form:

$$\bar{V}^{uu}_{i\alpha}(\mathbf{x},\xi,q) = \bar{G}^{uu}_{i\alpha}(\mathbf{x},\xi,q)\frac{\sum_{i=0}^{n}\sum_{j=0}^{m+n}M^*_{ij}q^{i+j}}{\sum_{i=0}^{n}\sum_{j=0}^{m+n}N^*_{ij}q^{i+j}}$$

$$= \bar{G}^{uu}_{i\alpha}(\mathbf{x},\xi,q)\frac{\sum_{i=0}^{m+2n}M_iq^i}{\sum_{i=0}^{m+2n}N_iq^i} = \bar{G}^{uu}_{i\alpha}(\mathbf{x},\xi,q)\frac{M_{m+2n}}{N_{m+2n}}\left[1+\frac{\sum_{i=0}^{m+2n-1}R_iq^i}{q^{m+2n}+\sum_{i=0}^{m+2n-1}P_iq^i}\right], \tag{32}$$

where

$$M^*_{ij} = a_i^d(h_1B_j+h_2A_j);\ N^*_{ij} = b_i^d(B_j-A_j), \tag{33}$$

$$\begin{cases} L_i = \sum_{j=1}^{i+1} L^*_{(j-1),(i-j+1)} & \text{if}\quad 0 \leqslant i \leqslant n, \\ L_i = \sum_{j=1}^{n+1} L^*_{(j-1),(i-j+1)} & \text{if}\quad n+1 \leqslant i \leqslant n+m, \\ L_i = \sum_{j=1-m-n+i}^{n+1} L^*_{(j-1),(i-j+1)} & \text{if}\quad n+m+1 \leqslant i \leqslant m+2n, \end{cases} \tag{34}$$

where $L = M$ or N and

$$R_i = \frac{M_i}{M_{m+2n}} - \frac{N_i}{N_{m+2n}};\ P_i = \frac{N_i}{N_{m+2n}}\quad (i = 0,\ldots,m+2n-1). \tag{35}$$

The inverse Laplace transform of $\bar{V}^{uu}_{i\alpha}(\mathbf{x},\xi,q)$ in eqn (32), using the convolution theorem, is:

$$V^{uu}_{i\alpha}(\mathbf{x},\xi,t) = \frac{M_{m+2n}}{N_{m+2n}}\left[G^{uu}_{i\alpha}(\mathbf{x},\xi,t)\right.$$

$$\left. + \int_0^t G^{uu}_{i\alpha}(\mathbf{x},\xi,s)\sum_{k=1}^{m+2n}\frac{\sum_{i=0}^{m+2n-1}R_i\alpha_k^i}{(m+2n)\alpha_k^{m+2n-1}+\sum_{i=1}^{m+2n-1}iP_i\alpha_k^{i-1}}\,\mathrm{e}^{(t-s)\alpha_k}\,\mathrm{d}s\right], \tag{36}$$

where α_k is the kth of the $m+2n$ (supposed) real and distinct roots of the equation

$$q^{m+2n}+\sum_{i=0}^{m+2n-1} P_i q^i = 0. \tag{37}$$

Since the values of the $V_{i\alpha}^{uu}(\mathbf{x},\boldsymbol{\xi},t)$ and $G_{i\alpha}^{uu}(\mathbf{x},\boldsymbol{\xi},t)$ coincide for $t = 0$, the ratio M_{m+2n}/N_{m+2n} in eqn (36) has to be equal to 1.

It is possible to show that this requirement is fulfilled when the elastic parameters G and K, leading to $G_{i\alpha}^{uu}(\mathbf{x},\boldsymbol{\xi},t)$, coincide with the parameters G_0 and K_0 of the rheological model.

Hence, eqn (36) takes the following final form:

$$V_{i\alpha}^{uu}(\mathbf{x},\boldsymbol{\xi},t) = G_{i\alpha}^{uu}(\mathbf{x},\boldsymbol{\xi},t) + \int_0^t G_{i\alpha}^{uu}(\mathbf{x},\boldsymbol{\xi},s)\sum_{k=1}^{m+2n}\frac{\sum_{i=0}^{m+2n-1} R_i\alpha_k^i}{(m+2n)\alpha_k^{m+2n-1}+\sum_{i=1}^{m+2n-1} iP_i\alpha_k^{i-1}}\,\mathrm{e}^{(t-s)\alpha_k}\,\mathrm{d}s. \tag{38}$$

Equation (38) can be reduced to a simpler form taking into account that the function $G_{i\alpha}^{uu}(\mathbf{x},\boldsymbol{\xi},t)$ vanishes for $t < 0$ and is equal to $G_{i\alpha}^{uu}(\mathbf{x},\boldsymbol{\xi})$ for $t \geqslant 0$. Then (38) takes the form:

$$V_{i\alpha}^{uu}(\mathbf{x},\boldsymbol{\xi},t) = G_{i\alpha}^{uu}(\mathbf{x},\boldsymbol{\xi})\left[1-\sum_{k=1}^{m+2n}\frac{\sum_{i=0}^{m+2n-1} R_i\alpha_k^i}{(m+2n)\alpha_k^{m+2n-1}+\sum_{i=1}^{m+2n-1} iP_i\alpha_k^{i-1}}\,\frac{1-\mathrm{e}^{t\alpha_k}}{\alpha_k}\right]. \tag{39}$$

Note that coefficients R_i, P_i and α_k in eqn (39) depend only on the parameters of the rheological model and on the distance between the loaded point and the point where the stress components are calculated; therefore, they have to be evaluated only once during the solution of the time-dependent problem.

(b) *Space and time function product form.* An alternative useful form of the viscoelastic solution $V_{i\alpha}^{uu}(\mathbf{x},\boldsymbol{\xi},t)$ can be found (using the same procedure as above) as the sum of products of functions of time and space respectively, i.e. in the form:

$$V_{i\alpha}^{uu}(\mathbf{x},\boldsymbol{\xi},t) = f_{i\alpha}^{uu}(\mathbf{x},\boldsymbol{\xi})v_f(t)+g_{i\alpha}^{uu}(\mathbf{x},\boldsymbol{\xi})v_g(t), \tag{40}$$

where $f_{i\alpha}^{uu}(\mathbf{x},\boldsymbol{\xi})$ and $g_{i\alpha}^{uu}(\mathbf{x},\boldsymbol{\xi})$ are the same functions given in Appendix B relevant to the corresponding elastic solutions. It can be seen that the functions $v_i(t)$ ($i = f, g$) are given by

$$v_i(t) = \frac{M_{m+2n}}{N_{m+2n}}\left[1-\sum_{k=1}^{m+2n}\frac{\sum_{i=0}^{m+2n-1} R_i\alpha_k^i}{(m+2n)\alpha_k^{m+2n-1}+\sum_{i=1}^{m+2n-1} iP_i\alpha_k^{i-1}}\,\frac{1-\mathrm{e}^{t\alpha_k}}{\alpha_k}\right], \tag{41}$$

where R_i and P_i are given by eqn (35) and M_i, N_i by eqn (34) being M_{ij}^* and N_{ij}^* defined by the expressions:

for $v(t)$:

$$\begin{cases} M_{ij}^* = a_i^d(3B_j-4A_j), & (42)\\ N_{ij}^* = b_i^d(B_j-A_j), & (43)\end{cases}$$

for $v_g(t)$:

Table 5. Coefficients M_{ij}^*, N_{ij}^*, M_i, N_i, h_1, h_2 for the evaluation of the viscoelastic fundamental solutions [in the elastic solution dependent form (39)] for the 3D and 2D plane strain cases

	M_{ij}^*	N_{ij}^*	M_i	N_i	h_1	h_2
$V_{i\alpha}^{uu}$	$a_i^d(h_1B_i+h_2A_i)$	$b_i^d(B_i-A_i)$	eqn (34)	eqn (34)	$3(f_{i\alpha}^{uu}+g_{i\alpha}^{uu})/G_{i\alpha}^{uu}$	$-4f_{i\alpha}^{uu}/G_{i\alpha}^{uu}$
$V_{i\alpha}^{pu}$	—	—	$h_1B_i-h_2A_i$	B_i-A_i	$(f_{i\alpha}^{pu}+g_{i\alpha}^{pu})/G_{i\alpha}^{pu}$	$-2f_{i\alpha}^{pu}/G_{i\alpha}^{pu}$
$V_{ij\alpha}^{\sigma u}$	—	—	$h_1B_i+h_2A_i$	B_i-A_i	$(f_{ij\alpha}^{\sigma u}+g_{ij\alpha}^{\sigma u})/G_{ij\alpha}^{\sigma u}$	$-2f_{ij\alpha}^{\sigma u}/G_{ij\alpha}^{\sigma u}$
$V_{i\alpha}^{up}$	—	—	$h_1B_i+h_2A_i$	B_i-A_i	$(f_{i\alpha}^{up}+g_{i\alpha}^{up})/G_{i\alpha}^{up}$	$-2f_{i\alpha}^{up}/G_{i\alpha}^{up}$
$V_{i\alpha}^{pp}$	$b_i^d(h_1A_i+h_2B_i)$	$a_i^d(B_i-A_i)$	eqn (34)	eqn (34)	$f_{i\alpha}^{pp}/G_{i\alpha}^{pp}$	$g_{i\alpha}^{pp}/G_{i\alpha}^{pp}$
$V_{ij\alpha}^{\sigma p}$	$b_i^d(h_1A_i+h_2B_i)$	$a_i^d(B_i-A_i)$	eqn (34)	eqn (34)	$f_{ij\alpha}^{\sigma p}/G_{ij\alpha}^{\sigma p}$	$g_{ij\alpha}^{\sigma p}/G_{ij\alpha}^{\sigma p}$
$V_{i\alpha\beta}^{u\sigma}$	—	—	$h_1B_i+h_2A_i$	B_i-A_i	$(f_{i\alpha\beta}^{u\sigma}+g_{i\alpha\beta}^{u\sigma})/G_{i\alpha\beta}^{u\sigma}$	$-2f_{i\alpha\beta}^{u\sigma}/G_{i\alpha\beta}^{u\sigma}$
$V_{i\alpha\beta}^{p\sigma}$	$b_i^d(h_1A_i+h_2B_i)$	$a_i^d(B_i-A_i)$	eqn (34)	eqn (34)	$f_{i\alpha\beta}^{p\sigma}/G_{i\alpha\beta}^{p\sigma}$	$g_{i\alpha\beta}^{p\sigma}/G_{i\alpha\beta}^{p\sigma}$
$V_{ij\alpha\beta}^{\sigma\sigma}$	$b_i^d(h_1A_i+h_2B_i)$	$a_i^d(B_i-A_i)$	eqn (34)	eqn (34)	$f_{ij\alpha\beta}^{\sigma\sigma}/G_{ij\alpha\beta}^{\sigma\sigma}$	$g_{ij\alpha\beta}^{\sigma\sigma}/G_{ij\alpha\beta}^{\sigma\sigma}$

Table 6. Coefficients M_{ij}^*, N_{ij}^*, M_i, N_i, h_1, h_2 for the evaluation of the viscoelastic fundamental solutions [in the elastic solution dependent form (39)] for the 2D plane strain cases

	M_{ij}^*	N_{ij}^*	M_i	N_i	h_1	h_2
$V_{i\alpha}^{uu}$	$a_i^d(h_1B_i+h_2A_i)$	$b_i^dB_j$	eqn (34)	eqn (34)	$(3f_{i\alpha}^{uu}+g_{i\alpha}^{uu})/G_{i\alpha}^{uu}$	$(g_{i\alpha}^{uu}-f_{i\alpha}^{uu})/G_{i\alpha}^{uu}$
$V_{i\alpha}^{pu}$	—	—	$h_1B_i+h_2A_i$	B_i	$(f_{i\alpha}^{pu}+g_{i\alpha}^{pu})/G_{i\alpha}^{pu}$	$(g_{i\alpha}^{pu}-f_{i\alpha}^{pu})/G_{i\alpha}^{pu}$
$V_{ij\alpha}^{\sigma u}$	—	—	$h_1B_i+h_2A_i$	B_i	$(f_{ij\alpha}^{\sigma u}+G_{ij\alpha}^{\sigma u})/G_{ij\alpha}^{\sigma u}$	$(g_{ij\alpha}^{\sigma u}-f_{ij\alpha}^{\sigma u})/G_{ij\alpha}^{\sigma u}$
$V_{i\alpha}^{up}$	—	—	$h_1B_i+h_2A_i$	B_i	$(f_{i\alpha}^{up}+g_{i\alpha}^{up})/G_{i\alpha}^{up}$	$(g_{i\alpha}^{up}-f_{i\alpha}^{up})/G_{i\alpha}^{up}$
$V_{i\alpha}^{pp}$	$b_i^d(h_1A_i+h_2B_i)$	$a_i^dB_i$	eqn (34)	eqn (34)	$(f_{i\alpha}^{pp}+g_{i\alpha}^{pp})/G_{i\alpha}^{pp}$	$g_{i\alpha}^{pp}/G_{i\alpha}^{pp}$
$V_{ij\alpha}^{\sigma p}$	$b_i^d(h_1A_i+h_2B_i)$	$a_i^dB_i$	eqn (34)	eqn (34)	$(f_{ij\alpha}^{\sigma p}+g_{ij\alpha}^{\sigma p})/G_{ij\alpha}^{\sigma p}$	$g_{ij\alpha}^{\sigma p}/G_{ij\alpha}^{\sigma p}$
$V_{i\alpha\beta}^{u\sigma}$	—	—	$h_1B_i+h_2A_i$	B_i	$(f_{i\alpha\beta}^{u\sigma}+g_{i\alpha\beta}^{u\sigma})/G_{i\alpha\beta}^{u\sigma}$	$(g_{i\alpha\beta}^{u\sigma}-f_{i\alpha\beta}^{u\sigma})/G_{i\alpha\beta}^{u\sigma}$
$V_{i\alpha\beta}^{p\sigma}$	$b_i^d(h_1A_i+h_2B_i)$	$a_i^dB_i$	eqn (34)	eqn (34)	$(f_{i\alpha\beta}^{p\sigma}+g_{i\alpha\beta}^{p\sigma})/G_{i\alpha\beta}^{p\sigma}$	$g_{i\alpha\beta}^{p\sigma}/G_{i\alpha\beta}^{p\sigma}$
$V_{ij\alpha\beta}^{\sigma\sigma}$	$b_i^d(h_1A_i+h_2B_i)$	$a_i^dB_i$	eqn (34)	eqn (34)	$(f_{ij\alpha\beta}^{\sigma\sigma}+g_{ij\alpha\beta}^{\sigma\sigma})/G_{ij\alpha\beta}^{\sigma\sigma}$	$g_{ij\alpha\beta}^{\sigma\sigma}/G_{ij\alpha\beta}^{\sigma\sigma}$

$$\begin{cases} M_{ij}^* = a_i^d B_i, & (44) \\ N_{ij}^* = b_i^d (B_i - A_i). & (45) \end{cases}$$

In other words, from the comparison of eqn (40) with eqn (B6) it appears that the time functions $v_f(t)$ and $v_g(t)$ for the $V_{i\alpha}^{uu}(\mathrm{x}, \xi, t)$ solution are correlated (by the correspondence principle) to the coefficients:

$$\frac{3-4v}{(1-v)G}, \quad \frac{1}{(1-v)G}.$$

5.2. *Viscoelastic solution* $V_{(..)}^{rs}$

(a) *Elastic solution dependent form.* Using the same technique as in Section 5.1(a) all the fundamental viscoelastic solutions can be derived in the following form, analogous to (39):

$$V_{(..)}^{rs} = G_{(..)}^{rs}\left[1 - \sum_{i=1}^{m+\beta n} \frac{\sum_{i=0}^{m+\beta n-1} R_i \alpha_k^i}{(m+\beta n)\alpha_k^{m+\beta n-1} + \sum_{i=1}^{m+\beta n-1} iP_i \alpha_k^{i-1}} \frac{1-e^{t\alpha_k}}{\alpha_k}\right], \quad r, s = u, p, \sigma, \quad (46)$$

where $\beta = 1$ for the solutions $V_{i\alpha}^{pu}$, $V_{i\alpha}^{up}$, $V_{ij\alpha}^{\sigma u}$, $V_{i\alpha\beta}^{u\sigma}$, and $\beta = 2$ for the solutions $V_{i\alpha}^{uu}$, $V_{i\alpha}^{pp}$, $V_{ij\alpha\beta}^{\sigma\sigma}$, $V_{ij\alpha}^{\sigma p}$, $V_{i\alpha\beta}^{p\sigma}$.

This can be made using the elastic solutions $G_{i\alpha}^{rs}(r, s = u, p, \sigma)$ given in Appendix B and the coefficients R_i, P_i given by eqn (35) where M_i, N_i (or M_{ij}^*, N_{ij}^*) are given in Table 5 (for the 3D and 2D plane strain cases) and in Table 6 (for the 2D plane stress case). The variables α_k are the roots of the equation:

Table 7. Coefficients M^*_{ij}, N^*_{ij} (or M_i, N_i) for the evaluation [by eqns (43), (35)] of the functions $v_f(t)$, $v_g(t)$ of the viscoelastic fundamental solutions [in the space and time function product form of the type eqn (42)] on the basis of the coefficients appearing in eqns (B6)–(B8), (B28)–(B30), (B47)–(B49) of the corresponding elastic fundamental solution

		M^*_{ij}	N^*_{ij}	M_i	N_i
3D and 2D (plane strain) cases	$\frac{3-4\nu}{(1-\nu)G}$	$a^d_i(3B_i-4A_i)$	$b^d_i(B_i-A_i)$	eqn (34)	eqn (34)
	$\frac{1}{(1-\nu)G}$	$a^d_iB_i$	$b^d_i(B_i-A_i)$	eqn (34)	eqn (34)
	$\frac{1-2\nu}{1-\nu}$	—	—	B_i-2A_i	B_i-A_i
	$\frac{1}{1-\nu}$	—	—	B_i	B_i-A_i
	$\frac{\nu G}{1-\nu}$	$b^d_iA_i$	$a^d_i(B_i-A_i)$	eqn (34)	eqn (34)
	$\frac{G}{1-\nu}$	$b^d_iB_i$	$a^d_i(B_i-A_i)$	eqn (34)	eqn (34)
2D (plane stress) case	$\frac{3-\nu}{G}$	$a^d_i(3B_i-A_i)$	$b^d_iB_i$	eqn (34)	eqn (34)
	$\frac{1+\nu}{G}$	$a^d_i(A_i+B_i)$	$b^d_iB_i$	eqn (34)	eqn (34)
	$1-\nu$	—	—	B_i-A_i	B_i
	$1+\nu$	—	—	B_i+A_i	B_i
	νG	$b^d_iA_i$	$a^d_iB_i$	eqn (34)	eqn (34)
	$G(1+\nu)$	$b^d_i(A_i+B_i)$	$a^d_iB_i$	eqn (34)	eqn (34)

$$q^{m+\beta n}+\sum_{i=0}^{m+\beta n-1}P_iq^i=0. \tag{47}$$

(b) *Space and time function product form.* Using the same technique as in Section 5.1(b) all the fundamental viscoelastic solutions can be written as the sum of products of functions of time and space, respectively, i.e. in the form:

$$V^{rs}_{(\cdot\cdot)}=f^{rs}_{(\cdot\cdot)}v_f(t)+g^{rs}_{(\cdot\cdot)}v_g(t), \tag{48}$$

where $f^{rs}_{(\cdot\cdot)}$ and $g^{rs}_{(\cdot\cdot)}$ are the same functions given in Appendix B relevant to the corresponding elastic solutions, and $v_i(t)$ ($i=f,g$) are given by:

$$v_i(t)=\frac{M_{m+2n}}{N_{m+2n}}\left[1-\sum_{k=1}^{m+\beta n}\frac{\sum_{i=0}^{m+\beta n-1}R_i\alpha_k^i}{(m+\beta n)\alpha_k^{m+\beta n-1}+\sum_{i=1}^{m+\beta n-1}iP_i\alpha_k^{i-1}}\,\frac{1-e^{t\alpha_k}}{\alpha_k}\right], \tag{41}$$

where $\beta=1$ for the solutions $V^{pu}_{i\alpha}$, $V^{up}_{i\alpha}$, $V^{\sigma u}_{ij\alpha}$, $V^{u\sigma}_{i\alpha\beta}$ and $\beta=2$ for the solutions $V^{uu}_{i\alpha}$, $V^{pp}_{i\alpha}$, $V^{\sigma\sigma}_{ij\alpha\beta}$, $V^{\sigma p}_{ij\alpha}$, $V^{p\sigma}_{i\alpha\beta}$.

In general, with reference to the generic viscoelastic fundamental solution the coefficients M^*_{ij}, N^*_{ij} (or M_i, N_i) required in order to define time functions $v_f(t)$, $v_g(t)$ of the form given by eqn (40) are correlated (through the correspondence principle) to the coefficients of the corresponding elastic solution in eqns (B6)–(B8), (B28)–(B30) and (B47)–(B49). This means that all the possible functions $v_f(t)$, $v_g(t)$ for all the viscoelastic fundamental solutions depends only on the six elastic coefficients relevant to the 3D and 2D plane strain case and the six coefficients relevant to the 2D plane stress case.

In Table 7 all the coefficients M^*_{ij}, N^*_{ij} (or M_i, N_i) for all such cases are given.

6. CONCLUSIONS

The fundamental solutions for a unit concentrated force (Kelvin) for a unit concentrated displacement discontinuity (Gebbia), and for a unit concentrated deformation of the linear viscoelastic 3D and 2D problems have been derived using the correspondence principle.

The following remarks can be made:

(1) using the same technique, the analogous fundamental solutions can be found in the visco-elasto-dynamic range;

(2) Kelvin's and Gebbia's fundamental solutions have proved their usefulness in the formulations of symmetric BIE in viscoelasticity (Carini *et al.*, 1991);

(3) using the fundamental solution for a unit concentrated deformation (besides those by Kelvin and Gebbia) it is possible to derive variational formulations of BIE even for visco-elastic–plastic problems.

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APPENDIX A: RECURSIVE FORMULA FOR THE GENERALIZED KELVIN AND MAXWELL MODELS

(a) *Generalized Kelvin–Voigt model*

To find the recursive formula for the model means finding the coefficients $a_i(n)$, $b_i(n)$ of the constitutive law:

$$\sum_0^n {}_i a_i(n)\sigma_n^{(i)} = \sum_0^n {}_i b_i(n)\varepsilon_{n-1}^{(i)}, \tag{A1}$$

relevant to the model of Fig. A1(b) with n patterns when the coefficients $a_i(n-1)$, $b_i(n-1)$ of the model of Fig. A1(a) with $n-1$ patterns are all known under the same stress $\sigma_n = \sigma_{n-1} = \sigma$. This will be obtained in two phases: (1) determining the recursive formula for the model of Fig. A1, without the elastic spring k_0; (2) modifying the recursive formula obtained in order to take into account the presence of spring k_0.

Phase 1. Let

$$\sum_0^{n-1} {}_i a_i^*(n-1)\sigma^{(i)} = \sum_0^{n-1} {}_i b_i^*(n-1)\varepsilon_{n-1}^{(i)} \tag{A2}$$

be the constitutive law relevant to the nonelastic part of the model of Fig. A2(a) (deleting the spring k_0). The corresponding constitutive law in the presence of the nth pattern will be (model of Fig. A2(b)):

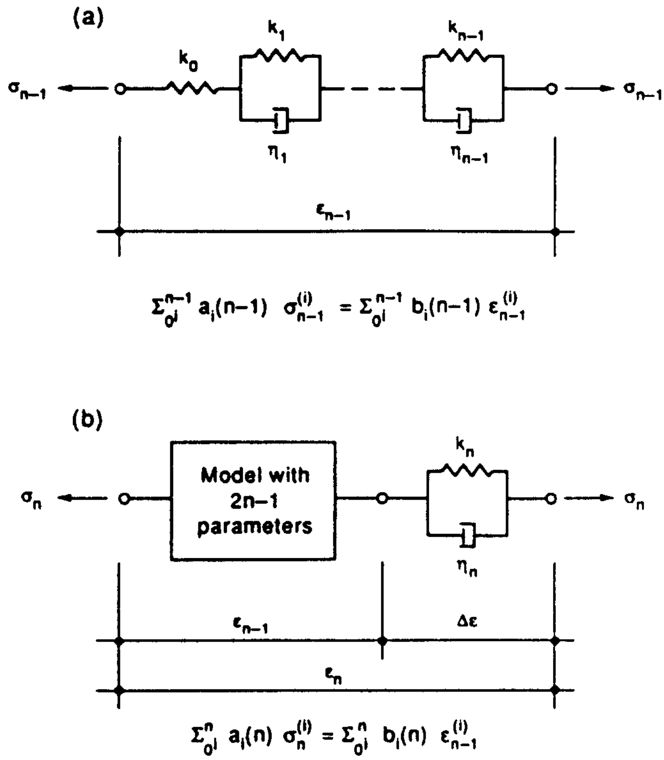

Fig. A1. (a) The generalized $2(n-1)+1$ Kelvin–Voigt model; (b) the "incremented" (by the nth pattern) $2n+1$ model.

$$\sum_{0}^{n}{}_i\, a_i^*(n)\sigma^{(i)} = \sum_{0}^{n}{}_i\, b_i^*(n)\varepsilon_n^{(i)} \tag{A3}$$

and the following relations will hold:

$$\begin{cases} \varepsilon_n = \varepsilon_{n-1} + \Delta\varepsilon, & \text{(A4)} \\ \sigma = k_n\Delta\varepsilon + \eta_n\Delta\dot{\varepsilon}, & \text{(A5)} \end{cases}$$

where

$$\Delta\dot{\varepsilon} = \dot{\varepsilon}_n - \dot{\varepsilon}_{n-1}. \tag{A6}$$

By substitution of eqn (A6) into (A5), the following relation is derived:

$$\sigma = k_n(\varepsilon_n - \varepsilon_{n-1}) + \eta_n(\dot{\varepsilon}_n - \dot{\varepsilon}_{n-1}). \tag{A7}$$

The Laplace transform of eqns (A2) and (A7) give:

$$\begin{cases} \sum_{0}^{n-1}{}_i\, a_i^*(n-1)q^i\bar{\sigma} = \sum_{0}^{n-1}{}_i\, b_i^*(n-1)q^i\bar{\varepsilon}_{n-1}, & \text{(A8)} \\ \bar{\sigma} = k_n(\bar{\varepsilon}_n - \bar{\varepsilon}_{n-1}) + \eta_n q(\bar{\varepsilon}_n - \bar{\varepsilon}_{n-1}). & \text{(A9)} \end{cases}$$

The pre-multiplication of both members of eqn (A9) by $b_i^*(n-1)q^i$, and the summation for $i = 0, \ldots, n-1$ taking into account eqn (A8) gives:

$$\sum_{0}^{n-1}{}_i\, [b_i^*(n-1) + k_n a_i^*(n-1)]q^i\bar{\sigma} + \sum_{0}^{n-1}{}_i\, \eta_n a_i^*(n-1)q^{i+1}\bar{\sigma} = \sum_{0}^{n-1}{}_i\, k_n b_i^*(n-1)q^i\bar{\varepsilon}_n + \sum_{0}^{n-1}{}_i\, \eta_n b_i^*(n-1)q^{i+1}\bar{\varepsilon}_n. \tag{A10}$$

The Laplace inverse transform of (A10) gives:

$$\sum_{0}^{n-1}{}_i\, [b_i^*(n-1) + k_n a_i^*(n-1)]\sigma^{(i)} + \sum_{0}^{n-1}{}_i\, \eta_n a_i^*(n-1)\sigma^{(i+1)} = \sum_{0}^{n-1}{}_i\, k_n b_i^*(n-1)\varepsilon_n^{(i)} + \sum_{0}^{n-1}{}_i\, \eta_n b_i^*(n-1)\varepsilon_n^{(i+1)} \tag{A11}$$

from which, by comparison with eqn (A3), and for $n \geq 1$:

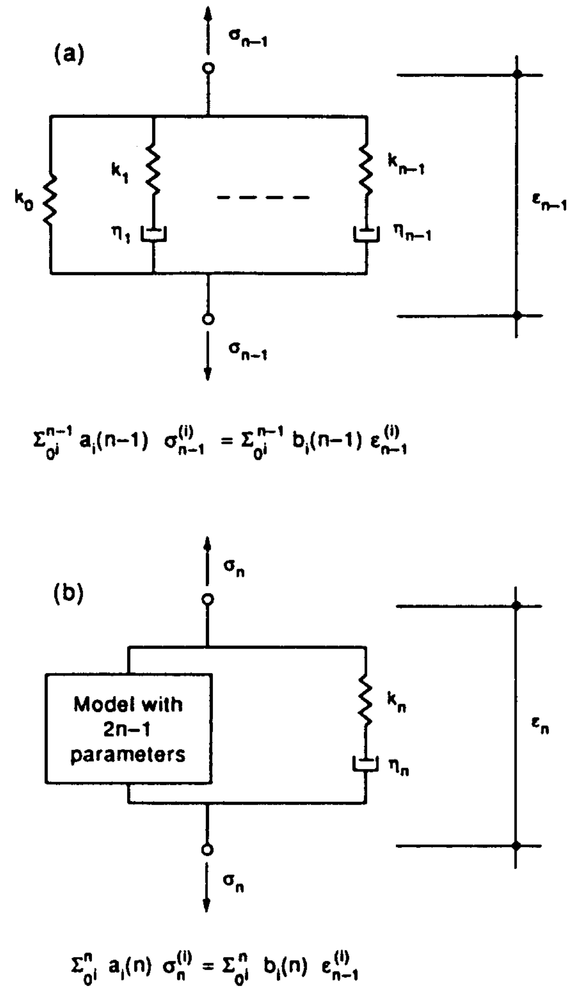

Fig. A2. (a) The generalized $2(n-1)+1$ Maxwell model; (b) the "incremented" (by the nth pattern) $2n+1$ model.

$$\begin{cases} a_i^*(n) = \eta_n a_{i-1}^*(n-1) + k_n a_i^*(n-1) + b_i^*(n-1), & \text{for} \quad i = 0, \ldots, n-1, \\ b_i^*(n) = \eta_n b_{i-1}^*(n-1) + k_n b_i^*(n-1), & \text{for} \quad i = 0, \ldots, n. \end{cases} \tag{A12}$$

This is the recursive formula of the model of Fig. A1 in the absence of the spring k_0 when, in order to avoid meaningful coefficients, the following positions are considered:

$$\begin{cases} a_{-1}^*(n-1) = a_{n-1}^*(n-1) = 0, \\ b_{-1}^*(n-1) = b_n^*(n-1) = 0, \\ b_0^*(0) = 1. \end{cases} \tag{A13}$$

Phase 2. The presence of the spring k_0 implies, with reference to the constitutive law (A1), the following relations:

$$\begin{cases} b_i(n) = b_i^*(n), \\ a_i(n) = a_i^*(n) + \dfrac{b_i^*(n)}{k_0}, \end{cases} \quad \text{for} \quad i = 0, 1, \ldots, n, \tag{A14}$$

under the obvious condition $a_n^*(n) = 0$ for $n \geq 1$.

The final recursive formula of the coefficients of the constitutive law (A1) for the model of Fig. A2(b), is given by the relations (A14) where the starred variables are given by relations (A12)–(A13).

Generalized Maxwell model

To find the recursive formula for the model means finding the coefficients $a_i(n), b_i(n)$ of the constitutive law:

$$\sum_0^n {}_i a_i(n)\sigma_n^{(i)} = \sum_0^n {}_i b_i(n)\varepsilon_{n-1}^{(i)} \tag{A15}$$

relevant to the model of Fig. A2(b) with n patterns when the coefficients $a_i(n-1), b_i(n-1)$ of the model of Fig. A2(a) with $n-1$ patterns are all known, being $\varepsilon_n = \varepsilon_{n-1} = \varepsilon$. By the adoption of the same technique as in Section (a) the following recursive formula of the coefficients of the constitutive law (A15) is obtained:

$$\begin{cases} a_i(n) = a_i^*(n), \\ b_i(n) = b_i^*(n) + k_0 a_i^*(n), \end{cases} \tag{A16}$$

where the starred variables are given by the following relations, for $n \geqslant 1$:

$$\begin{cases} a_i^*(n) = \dfrac{a_{i-1}^*(n-1)}{k_n} + \dfrac{a_i^*(n-1)}{\eta_n}, & \text{for} \quad i = 0, \ldots, n \\ b_i^*(n) = \dfrac{b_{i-1}^*(n-1)}{k_n} + \dfrac{b_i^*(n-1)}{\eta_n} + a_{i-1}^*(n-1), & \text{for} \quad i = 1, \ldots, n \end{cases} \tag{A17}$$

under the conditions:

$$\begin{cases} a_{-1}^*(n-1) = a_n^*(n-1) = 0, \\ b_0^*(n-1) = b_n^*(n-1) = 0, \\ a_0^*(0) = 1. \end{cases} \tag{A18}$$

APPENDIX B: FUNDAMENTAL SOLUTIONS IN LINEAR ELASTICITY AND THEIR CORRELATIONS

(a) *Kelvin fundamental solutions (for a unit concentrated force)*

These solutions refer to the case of a concentrated load (whose components are F_j, $j = 1, 2, 3$) in a given material point $\boldsymbol{\xi}$ of a 3D elastic homogeneous and isotropic unbounded continuum.

Let $G_{i\alpha}^{uu}(\mathbf{x}, \boldsymbol{\xi})$, $G_{i\alpha}^{pu}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{n})$ and $G_{ij\alpha}^{\sigma u}(\mathbf{x}, \boldsymbol{\xi})$ be the effects (in a given material point $\mathbf{x}$) of the load application in terms of the displacement component $u_i(\mathbf{x})$, of tractions $p_i(\mathbf{x})$ on the surface of normal $\mathbf{n}$ and of stress component σ_{ij} respectively, i.e. (using Greek and Latin symbols for "cause" and "effect" variables respectively):

$$\begin{cases} G_{i\alpha}^{uu}(\mathbf{x}, \boldsymbol{\xi})F_\alpha = u_i(\mathbf{x}), & \text{(B1)} \\ G_{i\alpha}^{pu}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{n})F_\alpha = \sigma_{ij}(\mathbf{x})n_j = p_i(\mathbf{x}), & \text{(B2)} \\ G_{ij\alpha}^{\sigma u}(\mathbf{x}, \boldsymbol{\xi})F_\alpha = \sigma_{ij}(\mathbf{x}). & \text{(B3)} \end{cases}$$

Obvious relations between the above Kelvin fundamental solutions are:

$$\begin{cases} G_{i\alpha}^{pu}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{n}) = G_{ij\alpha}^{\sigma u}(\mathbf{x}, \boldsymbol{\xi})n_j, & \text{(B4)} \\ G_{ij\alpha}^{\sigma u}(\mathbf{x}, \boldsymbol{\xi}) = G[G_{i\alpha,j}^{uu}(\mathbf{x}, \boldsymbol{\xi}) + G_{j\alpha,i}^{uu}(\mathbf{x}, \boldsymbol{\xi})] + \dfrac{2G\nu}{1-2\nu} G_{k\alpha,k}^{uu}(\mathbf{x}, \boldsymbol{\xi})\delta_{ij}. & \text{(B5)} \end{cases}$$

The full expressions of the above solutions are the following (Banerjee and Butterfield, 1981; Brebbia *et al.*, 1984):

$$G_{i\alpha}^{uu}(\mathbf{x}, \boldsymbol{\xi}) = \frac{(3-4\nu)}{(1-\nu)G} f_{i\alpha}^{uu}(\mathbf{x}, \boldsymbol{\xi}) + \frac{1}{(1-\nu)G} g_{i\alpha}^{uu}(\mathbf{x}, \boldsymbol{\xi}), \tag{B6}$$

$$G_{i\alpha}^{pu}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{n}) = \frac{1-2\nu}{1-\nu} f_{i\alpha}^{pu}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{n}) + \frac{1}{1-\nu} g_{i\alpha}^{pu}(\mathbf{x}, \boldsymbol{\xi}, \mathbf{n}), \tag{B7}$$

$$G_{ij\alpha}^{\sigma u}(\mathbf{x}, \boldsymbol{\xi}) = \frac{1-2\nu}{1-\nu} f_{ij\alpha}^{\sigma u}(\mathbf{x}, \boldsymbol{\xi}) + \frac{1}{1-\nu} g_{ij\alpha}^{\sigma u}(\mathbf{x}, \boldsymbol{\xi}), \tag{B8}$$

where

$$r_{,i} = \frac{x_i - \xi_i}{r}, \quad r_{,\alpha} = \frac{x_\alpha - \xi_\alpha}{r}, \quad r_{,n} = r_{,k}n_k \tag{B9}$$

and, for the 3D-case:

$$f_{i\alpha}^{uu}(\mathbf{x}, \boldsymbol{\xi}) = \frac{\delta_{i\alpha}}{16\pi r}, \tag{B10}$$

$$g_{i\alpha}^{uu}(\mathbf{x}, \boldsymbol{\xi}) = \frac{r_{,i}r_{,\alpha}}{16\pi r}, \tag{B11}$$

$$f^{pu}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{n}) = f^{\sigma u}_{ij\alpha}(\mathbf{x},\boldsymbol{\xi})n_j, \tag{B12}$$

$$g^{pu}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{n}) = g^{\sigma u}_{ij\alpha}(\mathbf{x},\boldsymbol{\xi})n_j, \tag{B13}$$

$$f^{\sigma u}_{ij\alpha}(\mathbf{x},\boldsymbol{\xi}) = -\frac{1}{8\pi r^2}(r_{,j}\delta_{\alpha i}+r_{,i}\delta_{j\alpha}-r_{,\alpha}\delta_{ij}), \tag{B14}$$

$$g^{\sigma u}_{ij\alpha}(\mathbf{x},\boldsymbol{\xi}) = -\frac{3}{8\pi r^2}r_{,i}r_{,j}r_{,\alpha}, \tag{B15}$$

while for the 2D-plane strain case:

$$f^{uu}_{i\alpha}(\mathbf{x},\boldsymbol{\xi}) = -\frac{1}{8\pi}\ln(r)\delta_{i\alpha}, \tag{B16}$$

$$g^{uu}_{i\alpha}(\mathbf{x},\boldsymbol{\xi}) = \frac{1}{8\pi}r_{,i}r_{,\alpha}, \tag{B17}$$

$$f^{pu}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{n}) = f^{\sigma u}_{ij\alpha}(\mathbf{x},\boldsymbol{\xi})n_j, \tag{B18}$$

$$g^{pu}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{n}) = g^{\sigma u}_{ij\alpha}(\mathbf{x},\boldsymbol{\xi})n_j, \tag{B19}$$

$$f^{\sigma u}_{ij\alpha}(\mathbf{x},\boldsymbol{\xi}) = -\frac{1}{4\pi r}(r_{,j}\delta_{\alpha i}+r_{,i}\delta_{j\alpha}-r_{,\alpha}\delta_{ij}), \tag{B20}$$

$$g^{\sigma u}_{ij\alpha}(\mathbf{x},\boldsymbol{\xi}) = -\frac{1}{2\pi r}r_{,i}r_{,j}r_{,\alpha}. \tag{B21}$$

Finally, for the 2D-plane stress case the same relations for the 2D-plane strain case hold, when the Poisson ratio v is substituted with:

$$v^* = \frac{v}{1+v}. \tag{B22}$$

(b) *Gebbia fundamental solutions (for a unit concentrated displacement discontinuity)*

These solutions refer to the case of a concentrated displacement discontinuity (whose components are D^l_j, $j = 1, 2, 3$) in a given material point $\boldsymbol{\xi}$ (crossing a surface Γ with normal $\mathbf{l}$) of a 3D elastic homogeneous and isotropic unbounded continuum.

Let $G^{up}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{l})$, $G^{pp}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{n},\mathbf{l})$ and $G^{\sigma p}_{ij\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{l})$ be the effects (in a given material point $\mathbf{x}$) of the load application in terms of the displacement component $u_i(\mathbf{x})$, of tractions $p_i(\mathbf{x})$ on the surface of normal $\mathbf{n}$ and of stress component $\sigma_{ij}(\mathbf{x})$ respectively, i.e. (using Greek and Latin symbols for "cause" and "effect" variables, respectively):

$$\begin{cases} G^{up}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{l})D^l_\alpha = u_i(\mathbf{x}), & \text{(B23)}\\ G^{pp}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{n},\mathbf{l})D^l_\alpha = \sigma_{ij}(\mathbf{x})n_j = p_i(\mathbf{x}), & \text{(B24)}\\ G^{\sigma p}_{ij\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{l})D^l_\alpha = \sigma_{ij}(\mathbf{x}). & \text{(B25)}\end{cases}$$

Obvious relations between the above Gebbia fundamental solutions are:

$$\begin{cases} G^{pp}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{n},\mathbf{l}) = G^{\sigma p}_{ij\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{l})n_j, & \text{(B26)}\\ G^{\sigma p}_{ij\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{l}) = G[G^{up}_{i\alpha,j}(\mathbf{x},\boldsymbol{\xi},\mathbf{l})+G^{up}_{j\alpha,i}(\mathbf{x},\boldsymbol{\xi},\mathbf{l})]+\dfrac{2Gv}{1-2v}G^{up}_{k\alpha,k}(\mathbf{x},\boldsymbol{\xi},\mathbf{l})\delta_{ij}. & \text{(B27)}\end{cases}$$

The full expressions of the above solutions are the following (Gebbia, 1881, 1902, 1904):

$$G^{up}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{l}) = \frac{1-2v}{1-v}f^{up}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{l})+\frac{1}{1-v}g^{up}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{l}), \tag{B28}$$

$$G^{pp}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{n},\mathbf{l}) = \frac{vG}{1-v}f^{pp}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{n},\mathbf{l})+\frac{G}{1-v}g^{pp}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{n},\mathbf{l}), \tag{B29}$$

$$G^{\sigma p}_{ij\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{l}) = \frac{vG}{1-v}f^{\sigma p}_{ij\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{l})+\frac{G}{1-v}g^{\sigma p}_{ij\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{l}), \tag{B30}$$

where for the 3D-case:

$$f^{up}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{l}) = f^{pu}_{\alpha i}(\boldsymbol{\xi},\mathbf{x},\mathbf{l}) = f^{\sigma u}_{\alpha ji}(\boldsymbol{\xi},\mathbf{x})l_j = \frac{1}{8\pi r^2}(r_{,j}\delta_{i\alpha}+r_{,\alpha}\delta_{ij}-r_{,i}\delta_{\alpha j})l_j, \tag{B31}$$

$$g^{up}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{l}) = g^{pu}_{\alpha i}(\boldsymbol{\xi},\mathbf{x},\mathbf{l}) = g^{\sigma u}_{\alpha ji}(\boldsymbol{\xi},\mathbf{x})l_j = \frac{3}{8\pi r^2}r_{,i}r_{,j}r_{,\alpha}l_j. \tag{B32}$$

$$f^{pp}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{n},\mathbf{l}) = f^{\sigma p}_{ij\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{l})n_j, \tag{B33}$$

$$g^{pp}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{n},\mathbf{l}) = g^{\sigma p}_{ij\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{l})n_j, \tag{B34}$$

$$f^{\sigma p}_{ij\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{l}) = \frac{-1}{4\pi r^2}(2\delta_{i\alpha}\delta_{kj}+2\delta_{ki}\delta_{\alpha j}-4\delta_{\alpha k}\delta_{ij}-3\delta_{i\alpha}r_{,j}r_{,k}-3\delta_{ki}r_{,\alpha}r_{,j}$$
$$-3\delta_{j\alpha}r_{,i}r_{,k}-3\delta_{kj}r_{,\alpha}r_{,i}+6\delta_{\alpha k}r_{,i}r_{,j}+6\delta_{ij}r_{,\alpha}r_{,k})l_k, \tag{B35}$$

$$g^{\sigma p}_{ij\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{l}) = \frac{1}{4\pi r^3}(\delta_{i\alpha}\delta_{kj}+\delta_{ki}\delta_{\alpha j}-\delta_{\alpha k}\delta_{ij}+3\delta_{\alpha k}r_{,i}r_{,j}+3\delta_{ij}r_{,k}r_{,\alpha}-15r_{,i}r_{,j}r_{,k}r_{,\alpha})l_k, \tag{B36}$$

while for the 2D-plane strain case:

$$f^{up}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{l}) = \frac{1}{4\pi r}(r_{,k}\delta_{i\alpha}+r_{,\alpha}\delta_{ki}-r_{,i}\delta_{\alpha k})l_k, \tag{B37}$$

$$g^{up}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{l}) = \frac{1}{2\pi r}r_{,i}r_{,\alpha}r_{,k}l_k, \tag{B38}$$

$$f^{pp}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{n},\mathbf{l}) = f^{\sigma p}_{ij\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{l})n_j = 0, \tag{B39}$$

$$g^{pp}_{i\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{n},\mathbf{l}) = g^{\sigma p}_{ij\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{l})n_j, \tag{B40}$$

$$g^{\sigma p}_{ij\alpha}(\mathbf{x},\boldsymbol{\xi},\mathbf{l}) = \frac{1}{2\pi r^2}(\delta_{i\alpha}\delta_{kj}+\delta_{ki}\delta_{\alpha j}-\delta_{\alpha k}\delta_{ij}+2\delta_{\alpha k}r_{,i}r_{,j}+2\delta_{ij}r_{,\alpha}r_{,k}-8r_{,i}r_{,j}r_{,\alpha}r_{,k})l_k. \tag{B41}$$

Finally, for the 2D-plane stress case the same relations for the 2D-plane strain case hold, when the Poisson ratio ν is substituted with ν^* given by eqn (B22).

(c) *Fundamental solutions (for a unit concentrated deformation)*

These solutions refer to the case of a concentrated deformation (whose components are Θ_{ij}, $i, j = 1, 2, 3$) in a given material point $\boldsymbol{\xi}$ (crossing a surface Γ with normal $\mathbf{l}$) of a 3D elastic homogeneous and isotropic unbounded continuum.

Let $G^{u\sigma}_{i\alpha\beta}(\mathbf{x},\boldsymbol{\xi})$, $G^{p\sigma}_{i\alpha\beta}(\mathbf{x},\boldsymbol{\xi},\mathbf{n})$ and $G^{\sigma\sigma}_{ij\alpha\beta}(\mathbf{x},\boldsymbol{\xi})$ the effects (at a given material point $\mathbf{x}$) of the load application in terms of the displacement component $u_i(\mathbf{x})$, of tractions $p_i(\mathbf{x})$ on the surface of normal $\mathbf{n}$ and of stress component $\sigma_{ij}(\mathbf{x})$ respectively, i.e. (using Greek and Latin symbols for "cause" and "effect" variables, respectively):

$$\begin{cases} G^{u\sigma}_{i\alpha\beta}(\mathbf{x},\boldsymbol{\xi})\Theta_{\alpha\beta} = u_i(\mathbf{x}), & \text{(B42)} \\ G^{p\sigma}_{i\alpha\beta}(\mathbf{x},\boldsymbol{\xi},\mathbf{n})\Theta_{\alpha\beta} = \sigma_{ij}(\mathbf{x})n_j = p_i(\mathbf{n}), & \text{(B43)} \\ G^{\sigma\sigma}_{ij\alpha\beta}(\mathbf{x},\boldsymbol{\xi})\Theta_{\alpha\beta} = \sigma_{ij}(\mathbf{x}). & \text{(B44)} \end{cases}$$

Obvious relations between the above fundamental solutions are:

$$\begin{cases} G^{p\sigma}_{i\alpha\beta}(\mathbf{x},\boldsymbol{\xi},\mathbf{n}) = G^{\sigma\sigma}_{ij\alpha\beta}(\mathbf{x},\boldsymbol{\xi})n_j, & \text{(B45)} \\ G^{\sigma\sigma}_{ij\alpha\beta}(\mathbf{x},\boldsymbol{\xi}) = G[G^{u\sigma}_{i\alpha\beta/j}(\mathbf{x},\boldsymbol{\xi})+G^{u\sigma}_{j\alpha\beta/i}(\mathbf{x},\boldsymbol{\xi})]+\dfrac{2G\nu}{1-2\nu}G^{u\sigma}_{k\alpha\beta/k}(\mathbf{x},\boldsymbol{\xi})\delta_{ij}. & \text{(B46)} \end{cases}$$

The full expressions of the above solutions are the following:

$$G^{u\sigma}_{i\alpha\beta}(\mathbf{x},\boldsymbol{\xi}) = \frac{1-2\nu}{1-\nu}f^{u\sigma}_{i\alpha\beta}(\mathbf{x},\boldsymbol{\xi})+\frac{1}{1-\nu}g^{u\sigma}_{i\alpha\beta}(\mathbf{x},\boldsymbol{\xi}), \tag{B47}$$

$$G^{p\sigma}_{i\alpha\beta}(\mathbf{x},\boldsymbol{\xi},\mathbf{n}) = \frac{\nu G}{1-\nu}f^{p\sigma}_{i\alpha\beta}(\mathbf{x},\boldsymbol{\xi},\mathbf{n})+\frac{G}{1-\nu}g^{p\sigma}_{i\alpha\beta}(\mathbf{x},\boldsymbol{\xi},\mathbf{n}), \tag{B48}$$

$$G^{\sigma\sigma}_{ij\alpha\beta}(\mathbf{x},\boldsymbol{\xi}) = \frac{\nu G}{1-\nu}f^{\sigma\sigma}_{ij\alpha\beta}(\mathbf{x},\boldsymbol{\xi})+\frac{G}{1-\nu}g^{\sigma\sigma}_{ij\alpha\beta}(\mathbf{x},\boldsymbol{\xi}), \tag{B49}$$

where for the 3D-case:

$$f^{u\sigma}_{i\alpha\beta}(\mathbf{x},\boldsymbol{\xi}) = f^{\sigma u}_{\alpha\beta i}(\boldsymbol{\xi},\mathbf{x}) = \frac{1}{8\pi r^2}(\delta_{i\alpha}r_{,\beta}+\delta_{\beta i}r_{,\alpha}-\delta_{\alpha\beta}r_{,i}), \tag{B50}$$

$$g^{u\sigma}_{i\alpha\beta}(\mathbf{x},\boldsymbol{\xi}) = g^{\sigma u}_{\alpha\beta i}(\boldsymbol{\xi},\mathbf{x}) = \frac{3}{8\pi r^2}r_{,\alpha}r_{,\beta}r_{,i}, \tag{B51}$$

$$f^{p\sigma}_{i\alpha\beta}(\mathbf{x},\boldsymbol{\xi},\mathbf{n}) = f^{\sigma\sigma}_{ij\alpha\beta}(\mathbf{x},\boldsymbol{\xi})n_j, \tag{B52}$$

$$g^{p\sigma}_{i\alpha\beta}(\mathbf{x},\boldsymbol{\xi},\mathbf{n}) = g^{\sigma\sigma}_{ij\alpha\beta}(\mathbf{x},\boldsymbol{\xi})n_j. \tag{B53}$$

$$f^{\sigma\sigma}_{ij\alpha\beta}(\mathbf{x},\xi) = -\frac{1}{4\pi r^3}(2\delta_{i\alpha}\delta_{\beta j}+2\delta_{j\alpha}\delta_{\beta i}-4\delta_{ij}\delta_{\alpha\beta}-3\delta_{i\alpha}r_{,\beta}r_{,j}-3\delta_{j\alpha}r_{,\beta}r_{,i}$$

$$-3\delta_{i\beta}r_{,\alpha}r_{,j}-3\delta_{j\beta}r_{,\alpha}r_{,i}+6\delta_{\alpha\beta}r_{,i}r_{,j}+6\delta_{ij}r_{,\alpha}r_{,\beta}). \quad \text{(B54)}$$

$$g^{\sigma\sigma}_{ij\alpha\beta}(\mathbf{x},\xi) = \frac{1}{4\pi r^3}(\delta_{i\alpha}\delta_{\beta j}+\delta_{j\alpha}\delta_{\beta i}-\delta_{\alpha\beta}\delta_{ij}+3\delta_{\alpha\beta}r_{,i}r_{,j}+3\delta_{ij}r_{,\beta}r_{,\alpha}-15r_{,i}r_{,j}r_{,\alpha}r_{,\beta}). \quad \text{(B55)}$$

and for the 2D-plane strain case:

$$f^{u\sigma}_{i\alpha\beta}(\mathbf{x},\xi) = f^{\sigma u}_{\alpha\beta i}(\xi,\mathbf{x}) = \frac{1}{4\pi r}(r_{,\beta}\delta_{i\alpha}+r_{,\alpha}\delta_{\beta i}+r_{,i}\delta_{\alpha\beta}). \quad \text{(B56)}$$

$$g^{u\sigma}_{i\alpha\beta}(\mathbf{x},\xi) = g^{\sigma u}_{\alpha\beta i}(\xi,\mathbf{x}) = \frac{1}{2\pi r}r_{,i}r_{,\alpha}r_{,\beta}. \quad \text{(B57)}$$

$$f^{p\sigma}_{i\alpha\beta}(\mathbf{x},\xi,\mathbf{n}) = f^{\sigma\sigma}_{ij\alpha\beta}(\mathbf{x},\xi)n_j = 0. \quad \text{(B58)}$$

$$g^{p\sigma}_{i\alpha\beta}(\mathbf{x},\xi,\mathbf{n}) = g^{\sigma\sigma}_{ij\alpha\beta}(\mathbf{x},\xi)n_j. \quad \text{(B59)}$$

$$g^{\sigma\sigma}_{ij\alpha\beta}(\mathbf{x},\xi) = \frac{1}{2\pi r^2}(\delta_{\alpha i}\delta_{j\beta}+\delta_{\alpha j}\delta_{i\beta}-\delta_{ij}\delta_{\alpha\beta}+2\delta_{ij}r_{,\alpha}r_{,\beta}+2\delta_{\alpha\beta}r_{,i}r_{,j}-8r_{,i}r_{,j}r_{,\alpha}r_{,\beta}). \quad \text{(B60)}$$

Finally, for the 2D-plane stress case the same relations for the 2D-plane strain case hold, when the Poisson ratio ν is substituted with ν^* given by eqn (B22).